BOUNDARY INTEGRAL EQUATIONS FOR THE BIHARMONIC DIRICHLET PROBLEM ON NONSMOOTH DOMAINS

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ABSTRACT. In this paper we study boundary integral formulations of the interior and exterior Dirichlet problem for the bi–Laplacian in a plane domain with a piecewise smooth boundary having corner points. The mapping properties of single and double layer biharmonic potentials, of the Calderon projections and the Poincaré-Steklov operators for such domains are analysed. We derive direct boundary integral equations equivalent to the variational formulation of the problem.

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1. INTRODUCTION

The paper is devoted to boundary integral methods for solving the Dirichlet problem of the biharmonic equation

$$\begin{aligned} &\Delta^2 u = 0 \quad \text{in } \Omega \subset \mathbb{R}^2 , \\ &u|_\Gamma = v|_\Gamma \ , \ \partial_n u|_\Gamma = \partial_n v|_\Gamma \ , \end{aligned} \tag{1.1}$$

where Ω is an interior or exterior domain bounded by a closed piecewise smooth curve Γ having corners and the Dirichlet data are the trace $(v|_\Gamma, \partial_n v|_\Gamma)$ of a function v belonging on a neighbourhood of Γ to the Sobolev space H^2. For the exterior problem one has to impose additionally a special behaviour of the solution at infinity.

The aim of the present paper is the study of direct boundary integral formulations which are equivalent to the variational solution of (1.1). As the main result we derive different systems of integral equations on Γ and describe their solvability conditions. To do so we introduce certain boundary integral operators for the bi-Laplacian and study mapping properties in the corresponding trace spaces of H^2-functions. As byproduct we are able to analyse the Steklov-Poincaré operators which map the Dirichlet data of biharmonic functions u to their Neumann data $(\Delta u|_\Gamma, \partial_n \Delta u|_\Gamma)$.

Among the different methods which exist for solving (1.1), integral equation methods play an important role, especially in connection with the boundary element method. For the interior problem and for sufficiently smooth boundary Γ such methods were investigated by several authors. Let us mention some results related to the contents of our paper. In [4] and [13] a system of direct boundary integral equations was studied which is closely connected with the

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system (6.10) of our approach. In [13] Fuglede derived necessary and sufficient conditions for the equivalence of these equations to (1.1) if the Dirichlet data are sufficiently smooth. A general approach of direct first kind integral equations for (1.1) can be performed using the results of Costabel and Wendland (see [6] and [12]). Based on the theory of pseudodifferential operators a complete description of the mapping properties of boundary integral operators, Calderon projections and Steklov-Poincaré operators can be obtained. This is mentioned in the paper of Costabel, Lusikka and Saranen [9], where approximation methods for solving the interior Dirichlet problem are studied, which are based on three different boundary integral formulations. Besides the equations coinciding with our systems (6.12) and (6.10) the authors consider also an indirect method which goes back to Hsiao and MacCamy [15] and is based on a single layer representation. This approach was extended by Costabel, Stephan and Wendland studying in [11], to our knowledge for the first time, boundary integral equations for the bi-Laplacian on a nonsmooth curve. The authors consider the related boundary value problem $grad\, u|_\Gamma = \mathbf{f}$ and obtain a system of two integral equations of the first kind with logarithmic principal part. Using Mellin techniques the continuity in Sobolev spaces and a Gårding inequality of the corresponding boundary integral operator are shown and the regularity of solutions is studied. Finally we mention the paper [2] of Bourlard which proposes a direct Galerkin BEM for solving the interior Dirichlet problem on a polygonal domain and obtains optimal convergence rates for special graded meshes. Many of the stability results for the Galerkin method appear also in our approach and we will comment these results at the corresponding places.

The paper is organized as follows. In Section 2 we consider the space of Dirichlet data of H^2-functions and the space of Neumann data of H^2-functions u with $\Delta^2 u \in L^2$. In Section 3 we introduce the biharmonic potentials and their traces, the boundary integral operators. We investigate mapping properties with respect to the trace spaces, the jump relations of the potentials and prove the Gårding inequality for the single layer potential operator. In Section 4 the behaviour at infinity for solutions of the exterior Dirichlet problem is specified and we prove representation formulas for the variational solutions of (1.1). This allows to represent the Calderon projections via boundary integral operators. The special structure of these projections is used in Section 5 to analyse the Steklov–Poincaré operators for biharmonic functions. We remark that in [17] a fast method for solving the interior Dirichlet problem (1.1) on convex polygonal domains is developed based on boundary reduction and mapping properties of Steklov–Poincaré operators. In the last Section we derive systems of integral equations for solving (1.1), partially new even for smooth Γ, and study the solvability of these equations.

To conclude the introduction we briefly comment some topics not treated in this paper. We do not consider the approximate solution of the integral equations. The convergence of Galerkin and certain collocation methods for the strongly elliptic system (6.6) is rather clear, whereas the stability of approximation methods for solving the other systems seems to be open. To get error estimates one has to know the regularity of the corresponding solutions. This topic and also the continuity of boundary integral operators in other than the energy norms we do not study because of the lack of space. Since we are dealing with direct methods some regularity results can be derived from the known singularities of the solutions of the Dirichlet problem (see [1]). On the other hand, the calculus of Mellin operators provides a useful tool in this direction. A more interesting problem not treated is the analysis of direct integral methods for the biharmonic equation with other boundary conditions. The application of our methods to this problem will be considered in a forthcoming paper (see Remark 5.3).

For the following let Γ be a simple closed curve in the plane (x_1, x_2) of the form

$$\Gamma = \bigcup_{i=1}^{n} \Gamma_i \,,$$

where Γ_i are of the class C^3 and adjacent arcs Γ_i form corners with angles different from 0 and 2π. The interior of Γ we denote by Ω_1, the exterior $\mathbb{R}^2 \backslash \overline{\Omega}_1$ by Ω_2, and let the unit normal n on Γ be directed into Ω_2. The differentiation with respect to n is denoted by ∂_n. The starting point of our analysis is

Lemma 2.1. (Jakovlev [16]). *Let $u \in H^2(\Omega_1)$. Then*

$$u|_{\Gamma_i} \in H^{3/2}(\Gamma_i)\,, \quad \partial_n u|_{\Gamma_i} \in H^{1/2}(\Gamma_i)\,,$$
$$u|_{\Gamma} \in H^1(\Gamma)\,, \quad \left.\frac{\partial u}{\partial x_1}\right|_{\Gamma}, \left.\frac{\partial u}{\partial x_2}\right|_{\Gamma} \in H^{1/2}(\Gamma)$$

and there exists a constant $c > 0$, not depending on u, such that

$$\sum_{i=1}^{n} \left(\|u\|_{H^{3/2}(\Gamma_i)} + \|\partial_n u\|_{H^{1/2}(\Gamma_i)} \right) + \left\| \frac{\partial u}{\partial x_1} \right\|_{H^{1/2}(\Gamma)} + \left\| \frac{\partial u}{\partial x_2} \right\|_{H^{1/2}(\Gamma)} \le c \, \|u\|_{H^2(\Omega)} \,.$$

If the projections of the normal n onto the x_1– and x_2–axis are denoted by n_1 and n_2, respectively, then

$$\left.\frac{\partial u}{\partial x_1}\right|_{\Gamma} = n_1 \, \partial_n u - n_2 \, \partial_s u \,, \quad \left.\frac{\partial u}{\partial x_2}\right|_{\Gamma} = n_2 \, \partial_n u + n_1 \, \partial_s u \,,$$

where $\partial_s u$ denotes the differentiation with respect to the arc length s. In the sequel we identify functions on Γ with periodic functions depending on s and write $\partial_s u = u'$. It is well known that for $|t| \le 1$ the Sobolev spaces $H^t(\Gamma)$ can be identified with the corresponding periodic Sobolev spaces.

Note that $n_1(s)$ and $n_2(s)$ as well as $\partial_n \varphi|_\Gamma$ and $\partial_s \varphi|_\Gamma$ for smooth $\varphi \in C_0^\infty(\mathbb{R}^2)$ are piecewise functions of the class C^2 and C^3, respectively, with jumps at the corner points. Let us introduce the trace space

$$V(\Gamma) = \left\{ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \,:\, u_1 \in H^1(\Gamma),\; n_1 u_2 - n_2 u_1' \in H^{1/2}(\Gamma),\; n_2 u_2 + n_1 u_1' \in H^{1/2}(\Gamma) \right\}$$

equipped with the canonical norm and define the generalized trace

$$\gamma u := \begin{pmatrix} u|_\Gamma \\ \partial_n u|_\Gamma \end{pmatrix} .$$

Lemma 2.2. (Jakovlev [16]). *The linear mapping*

$$\gamma : H^2_{loc}(\mathbb{R}^2) \to V(\Gamma)$$

is continuous and has a continuous right inverse

$$\gamma^- : V(\Gamma) \to H^2_{loc}(\mathbb{R}^2) \,.$$

In particular, γ maps $C_0^\infty(\mathbb{R}^2)$ onto a dense subspace of $V(\Gamma)$.

Let us describe the dual space of $V(\Gamma)$. We introduce the duality form

$$\left[\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right] := -\langle v_1, u_1 \rangle_\Gamma + \langle v_2, u_2 \rangle_\Gamma \,, \tag{2.1}$$

where $\langle \cdot, \cdot \rangle_\Gamma$ denotes the extension of the usual L^2–scalar product on Γ.

Lemma 2.3. *The vector* $\binom{v_1}{v_2}$ *belongs to* $(V(\Gamma))'$ *iff there exist* $z_1, z_2 \in H^{-1/2}(\Gamma)$ *such that the equations*

$$\begin{aligned} v_2 &= n_1 z_1 + n_2 z_2 \,, \\ \langle \varphi|_\Gamma, v_1 \rangle_\Gamma &= \langle \partial_s \varphi|_\Gamma, n_2 z_1 - n_1 z_2 \rangle_\Gamma \,, \qquad \forall\, \varphi \in C_0^\infty(\mathbb{R}^2) \,, \end{aligned}$$

are satisfied.

The trace $\gamma u \in V(\Gamma)$ will be called the *Dirichlet datum* of $u \in H^2_{loc}(\mathbb{R}^2)$ on Γ. Now we define the *Neumann datum.* We introduce the space

$$H^2(\Omega_1, \Delta^2) = \{u \in H^2(\Omega_1) : \Delta^2 u \in L^2(\Omega_1)\}$$

with the graph norm.

Lemma 2.4. $C^\infty(\overline{\Omega}_1)$ *is dense in* $H^2(\Omega_1, \Delta^2)$.
The proof is based on the same arguments as the proof for the case $H^1(\Omega_1, \Delta)$ given in the book of Grisvard [14].

Lemma 2.5. *Let* $u \in H^2(\Omega_1, \Delta^2)$. *Then the mapping*

$$\delta u \,:\, \psi \to [\delta u, \psi] := \int\limits_{\Omega_1} (\Delta u\, \Delta(\gamma^- \psi) - \gamma^- \psi\, \Delta^2 u) dx \tag{2.2}$$

is a continuous linear functional on $V(\Gamma)$ *that coincides for sufficiently smooth* u *with the functional*

$$\delta u := \begin{pmatrix} \partial_n \Delta u|_\Gamma \\ \Delta u|_\Gamma \end{pmatrix} .$$

Moreover, the mapping

$$\delta : H^2(\Omega_1, \Delta^2) \to (V(\Gamma))' \tag{2.3}$$

is continuous.
Proof. The first Green formula

$$\int\limits_{\Omega_1} (\Delta u\, \Delta v - v\, \Delta^2 u) dx = \int\limits_\Gamma (\Delta u\, \partial_n v - v\, \partial_n \Delta u)\, ds$$

is valid for all $u \in H^4(\Omega_1)$, $v \in H^2(\Omega_1)$ (see [5]). Hence for sufficiently smooth u

$$\begin{aligned} \big|[\delta u, \psi]\big| &\leq \|\Delta u\|_{L^2(\Omega_1)} \|\Delta(\gamma^- \psi)\|_{L^2(\Omega_1)} + \|\Delta^2 u\|_{L^2(\Omega_1)} \|\gamma^- \psi\|_{L^2(\Omega_1)} \\ &\leq \|u\|_{H^2(\Omega_1, \Delta^2)} \|\gamma^- \psi\|_{H^2(\Omega_1)} \,. \end{aligned}$$

From Lemmas 2.2 and 2.4 the assertion follows by continuity. ∎

Corollary 2.1. *For* $u, v \in H^2(\Omega_1, \Delta^2)$ *the second Green formula*

$$\int\limits_{\Omega_1} (v\, \Delta^2 u - u\, \Delta^2 v) dx = [\delta v, \gamma u] - [\delta u, \gamma v]$$

holds. If $u \in H^2(\Omega_1)$ *solves the biharmonic equation* $\Delta^2 u = 0$ *then*

$$[\delta u, \gamma u] \geq 0 \,.$$

The construction of the Neumann data δu is standard, for second order equations we refer to [14] and [7], for the biharmonic equation a similar construction is given in [2]. We note that the definition of δu is based on the bilinear form

$$a(u, v) := \int\limits_{\Omega_1} \Delta u\, \Delta v\, dx \;,$$

corresponding to the variational solution of the Dirichlet problem

$$\Delta^2 u = f \quad \text{in } \Omega_1 \,, \qquad \gamma u = \psi \,, \tag{2.4}$$

with $f \in L^2(\Omega_1)$, $\psi \in V(\Gamma)$. Since $a(u,u)^{1/2}$ is an equivalent norm on $H_0^2(\Omega_1)$ (see [5]) we derive by using Lemma 2.2 the unique solvability of (2.4) in variational sense.

Lemma 2.6. *The Dirichlet problem* (2.4) *has for any* $f \in L^2(\Omega_1)$, $\psi \in V(\Gamma)$ *a unique solution* $u \in H^2(\Omega_1, \Delta^2)$. *The solution operator*

$$T : L^2(\Omega_1) \times V(\Gamma) \to H^2(\Omega_1, \Delta^2) \tag{2.5}$$

is continuous.

Now we can prove

Lemma 2.7. δ *maps* $C_0^\infty(\mathbb{R}^2)$ *onto a dense subspace of* $(V(\Gamma))'$.
Proof. Assume that for some $\psi \in V(\Gamma)$ it holds

$$[\delta\varphi, \psi] = 0 \tag{2.6}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^2)$. Due to Lemma 2.6 the boundary value ψ and an arbitrary $f \in L^2(\Omega_1)$ lead to solutions $T(0,\psi)\,, T(f,0) \in H^2(\Omega_1, \Delta^2)$ of the corresponding Dirichlet problems. Applying Corollary 2.1 we obtain

$$\begin{aligned}
[\delta T(f,0), \psi] &= [\delta T(f,0), \gamma T(0,\psi)] - [\delta T(0,\psi), \gamma T(f,0)] \\
&= \int\limits_{\Omega_1} \left(T(f,0)\,\Delta^2 T(0,\psi) - T(0,\psi)\,\Delta^2 T(f,0)\right) dx \\
&= -\int\limits_{\Omega_1} f\,T(0,\psi)\,dx\,.
\end{aligned}$$

From Lemma 2.4 we conclude that (2.6) holds even for $\varphi = T(f,0) \in H^2(\Omega_1, \Delta^2)$, such that

$$\int\limits_{\Omega_1} f\,T(0,\psi)\,dx = 0 \quad \text{for all } \; f \in L^2(\Omega_1)\,.$$

Thus $T(0,\psi) = 0$ and the relation $\psi = \gamma T(0,\psi) = 0$ shows that $\delta(C_0^\infty(\mathbb{R}^2))$ is dense in $(V(\Gamma))'$. ∎

In the sequel we consider also the Dirichlet problem in the exterior domain Ω_2. Besides the Dirichlet datum we have therefore to define the Neumann datum of functions given outside of Ω_1. Let $\widetilde{\Omega}$ be a domain containing $\overline{\Omega}_1$ and let $u \in H^2(\widetilde{\Omega}\backslash\Omega_1, \Delta^2)$. For $v \in H^2(\widetilde{\Omega}\backslash\Omega_1)$ we define

$$[\delta u, \gamma v] = \int\limits_{\widetilde{\Omega}\backslash\Omega_1} (\varphi v\,\Delta^2 u - \Delta(\varphi v)\,\Delta u) dx\,,$$

where $\varphi \in C_0^\infty(\widetilde{\Omega})$ with $\varphi \equiv 1$ on a neighbourhood of $\overline{\Omega}_1$. It is clear that the definition of δ does not depend on φ. Moreover, it ensures that for $\varphi \in C_0^\infty(\mathbb{R}^2)$ there holds

$$\delta(\varphi|_{\Omega_2}) = \delta(\varphi|_{\Omega_1})\,.$$

In the following the pair of Dirichlet and Neumann data $(\gamma u, \delta u)$ will be called *Cauchy data* of u.

3. Boundary integral operators for the bi-Laplacian

Here we follow a method described in Costabel [7] for the study of boundary integral operators for second order equations on Lipschitz domains. The boundary integral operators for the bi–Laplacian Δ^2 are based on the fundamental solution

$$G(x,y) := \frac{1}{8\pi}|x-y|^2 \ln|x-y|\,, \quad x,y \in \mathbb{R}^2\,,$$

satisfying

$$\Delta_y^2 G(x,y) = \Delta_x^2 G(x,y) = \delta(x-y)\,.$$

It is well known that the operator

$$Gu(x) := \langle G(x,\cdot), u\rangle_{\mathbb{R}^2}$$

is the two–sided inverse of Δ^2 on the space of compactly supported distributions on $\mathbb{R}^2$ and represents a pseudodifferential operator of order -4, i.e.

$$G : H^s_{comp}(\mathbb{R}^2) \to H^{s+4}_{loc}(\mathbb{R}^2)\,, \quad s \in \mathbb{R}\,, \tag{3.1}$$

is continuous. Furthermore

$$\Delta_y G(x,y) = \Delta_x G(x,y) = \frac{1}{2\pi}\ln|x-y| + \frac{1}{2\pi}\,. \tag{3.2}$$

We have the following representation formula.

Lemma 3.1. *Let $u \in L^2(\mathbb{R}^2)$ be a function with compact support such that the restrictions $u|_{\Omega_1} \in H^2(\Omega_1)$, $u|_{\Omega_2} \in H^2_{comp}(\Omega_2)$ and $f = \Delta^2 u|_{\mathbb{R}^2\setminus\Gamma} \in L^2(\mathbb{R}^2)$. Then for $x \in \mathbb{R}^2\setminus\Gamma$ it holds*

$$u(x) = Gf(x) - [\{\delta u\}, \gamma G(x,\cdot)] + [\delta G(x,\cdot), \{\gamma u\}]\,,$$

where

$$\{\gamma u\} := \gamma(u|_{\Omega_2}) - \gamma(u|_{\Omega_1})\;,\; \{\delta u\} := \delta(u|_{\Omega_2}) - \delta(u|_{\Omega_1})\,,$$

denote the jumps across Γ.

The proof follows immediately from the second Green formula (Corollary 2.1) and the known representation formula for sufficiently smooth functions applied in a small ball enclosing the point x.

Next we define the biharmonic layer potentials for $x \in \mathbb{R}^2\setminus\Gamma$ as

$$\begin{aligned} \mathcal{K}_0\chi(x) &:= [\chi, \gamma G(x,\cdot)]\,, \qquad \chi \in (V(\Gamma))'\,, \\ \mathcal{K}_1\psi(x) &:= [\delta G(x,\cdot), \psi]\,, \qquad \psi \in V(\Gamma)\,, \end{aligned} \tag{3.3}$$

and the boundary integral operators

$$\begin{aligned} \mathcal{A}\chi &:= 2\,\gamma\mathcal{K}_0\chi \qquad\;\;,\;\; \mathcal{B}\chi := 2\,\delta(\mathcal{K}_0\chi|_{\Omega_1})\,, \\ \mathcal{C}\psi &:= 2\,\gamma(\mathcal{K}_1\psi|_{\Omega_1})\,,\; \mathcal{D}\psi := -2\,\delta(\mathcal{K}_1\psi|_{\Omega_1})\,. \end{aligned} \tag{3.4}$$

Lemma 3.2. *The mappings*

$$\begin{aligned} \mathcal{K}_0 : (V(\Gamma))' &\to H^2_{loc}(\mathbb{R}^2)\,, \quad \mathcal{K}_1 : V(\Gamma) \to H^2(\Omega_1)\,, \\ \mathcal{A} : (V(\Gamma))' &\to V(\Gamma)\,, \quad \mathcal{B} : (V(\Gamma))' \to (V(\Gamma))'\,, \quad \mathcal{C} : V(\Gamma) \to V(\Gamma) \end{aligned}$$

are continuous and

$$\mathcal{D}\psi = 0\,, \quad \psi \in V(\Gamma)\,.$$

Proof. Because of

$$\mathcal{K}_0\chi(x) = \langle G(x,\cdot), \gamma'\chi\rangle_{\mathbb{R}^2}$$

we can write

$$\mathcal{K}_0\chi = G\gamma'\chi\,. \tag{3.5}$$

The adjoint of the trace map $\gamma' : (V(\Gamma))' \to H^{-2}_{comp}(\mathbb{R}^2)$ is continuous, therefore the assertion for $\mathcal{K}_0$ follows from (3.1).

Due to Lemma 3.1 the solution $u = T(0,\psi)$ of the Dirichlet problem (2.4) can be represented in the form

$$T(0,\psi) = \mathcal{K}_0\delta T(0,\psi) - \mathcal{K}_1\psi\,,$$

such that from Lemmas 2.5 and 2.6 we derive

$$\|\mathcal{K}_1\psi\|_{H^2(\Omega_1)} \le c\,\|\psi\|_{V(\Gamma)}\,.$$

Now the mapping properties of $\mathcal{A}$ and $\mathcal{C}$ are a simple consequence of Lemma 2.1. The boundedness of $\mathcal{B}$ follows from Lemma 2.5 since $\Delta^2\mathcal{K}_0\chi = 0$ in Ω_1.

For $\psi = \binom{v_1}{v_2} \in V(\Gamma)$ we get from (2.1) and (3.2) the representation

$$\mathcal{K}_1\psi(x) = -\frac{1}{2\pi}\int_\Gamma v_1(y)\,\partial_{n_y}\ln|x-y|ds_y + \frac{1}{2\pi}\int_\Gamma v_2(y)\,(\ln|x-y|+1)ds_y\,. \tag{3.6}$$

Hence, $\mathcal{K}_1\psi \in H^2(\Omega_1, \Delta^2)$ is a harmonic function and for any $\varphi \in V(\Gamma)$

$$[\mathcal{D}\psi, \varphi] = 2\int_{\Omega_1}(\gamma^-\varphi\,\Delta^2\mathcal{K}_1\psi - \Delta(\gamma^-\varphi)\,\Delta\mathcal{K}_1\psi)\,dx = 0\,. \quad \blacksquare$$

The layer potentials provide the following jump relations:

Lemma 3.3.

$$\begin{array}{lll} \{\gamma\mathcal{K}_0\chi\} = 0\,, & \{\delta\mathcal{K}_0\chi\} = -\chi & \textit{for all}\ \ \chi \in (V(\Gamma))'\,, \\ \{\gamma\mathcal{K}_1\psi\} = \psi\,, & \{\delta\mathcal{K}_1\psi\} = 0 & \textit{for all}\ \ \psi \in V(\Gamma)\,. \end{array}$$

Proof. Since $u = \mathcal{K}_0\chi \in H^2_{loc}(\mathbb{R}^2)$ we have $\gamma(u|_{\Omega_1}) = \gamma(u|_{\Omega_2})$.
Further, from (3.5) we obtain $\Delta^2 u = \gamma'\chi$ in distributional sense, i.e.

$$\int_{\mathbb{R}^2} u\,\Delta^2\varphi dx = \langle\gamma'\chi, \varphi\rangle_{\mathbb{R}^2} = [\chi, \gamma\varphi]$$

for any $\varphi \in C_0^\infty(\mathbb{R}^2)$. On the other hand

$$\begin{aligned} \int_{\Omega_1} u\,\Delta^2\varphi dx &= \int_{\Omega_1}\Delta u\,\Delta\varphi dx - [\delta\varphi, \gamma u] = [\delta(u|_{\Omega_1}), \gamma\varphi] - [\delta\varphi, \gamma u]\,, \\ \int_{\Omega_2} u\,\Delta^2\varphi dx &= -[\delta(u|_{\Omega_2}), \gamma\varphi] + [\delta\varphi, \gamma u]\,. \end{aligned}$$

Thus

$$[\chi, \gamma\varphi] = -[\delta(\mathcal{K}_0\chi|_{\Omega_2}) - \delta(\mathcal{K}_0\chi|_{\Omega_1}), \gamma\varphi]\,, \quad \forall\,\varphi \in C_0^\infty(\mathbb{R}^2)\,.$$

Let now $u = \mathcal{K}_1\psi$, $\psi = \binom{v_1}{v_2} \in V(\Gamma)$. From (3.6) and the jump relations of the harmonic potentials (proved for example in [10] for the more general case $(v_1, v_2) \in H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)$) we obtain

$$u|_{\Omega_2} - u|_{\Omega_1} = v_1 \quad , \quad \partial_n u|_{\Omega_2} - \partial_n u|_{\Omega_1} = v_2\,. \quad \blacksquare$$

Now we consider the adjoints of the boundary integral operators with respect to the duality form (2.1). Here and in the following *Id* denotes the identity mapping in the spaces $V(\Gamma)$, $(V(\Gamma))'$ or $V(\Gamma)\times(V(\Gamma))'$.

Corollary 3.1. *There holds* $\mathcal{A} = \mathcal{A}'$ *and* $\mathcal{B}' = \mathcal{C} + 2\,Id$.

Proof. The assertion follows immediately from the symmetry of the kernel function G and the jump relations, for example:

$$\begin{aligned} [\mathcal{B}\chi, \psi] &= \Big[\delta(\mathcal{K}_0\chi|_{\Omega_1}) + \delta(\mathcal{K}_0\chi|_{\Omega_2}) + \chi, \psi\Big] \\ &= \Big\langle G\gamma'\chi|_{\Omega_1} + G\gamma'\chi|_{\Omega_2}, \delta'\psi\Big\rangle_{\mathbb{R}^2} + [\chi, \psi]\,, \end{aligned}$$

where $\delta'\psi$ denotes the compactly supported distribution on $\mathbb{R}^2$ defined by

$$\langle\varphi, \delta'\psi\rangle_{\mathbb{R}^2} = [\delta\varphi, \psi] \quad \text{for all}\ \ \varphi \in C_0^\infty(\mathbb{R}^2)\,.$$

Since the jump relation yields for $u = \mathcal{K}_1\psi$

$$\int\limits_{\mathbb{R}^2} u\,\Delta^2\varphi dx = [\delta\varphi, \psi] \quad \text{for all} \quad \varphi \in C_0^\infty(\mathbb{R}^2)$$

we have $\Delta^2 u = \delta'\psi$ in distributional sense and

$$\mathcal{K}_1\psi = G\delta'\psi\,.$$

Hence

$$\begin{aligned}[\mathcal{B}\chi, \psi] &= \langle G\gamma'\chi, \delta'\psi\rangle_{\mathbb{R}^2} + [\chi, \psi] = \langle \gamma'\chi, G\delta'\psi|_{\Omega_1} + G\delta'\psi|_{\Omega_2}\rangle_{\mathbb{R}^2} + [\chi, \psi] \\ &= [\chi, \gamma(\mathcal{K}_1\psi|_{\Omega_1}) + \gamma(\mathcal{K}_1\psi|_{\Omega_2})] + [\chi, \psi] = [\chi, 2\gamma(\mathcal{K}_1\psi|_{\Omega_1}) + \psi] + [\chi, \psi] \\ &= [\chi, \mathcal{C}\psi] + 2[\chi, \psi]\,. \qquad \blacksquare\end{aligned}$$

Let us introduce the operator

$$\mathcal{W} := Id + \mathcal{C}\,. \tag{3.7}$$

Then

$$\mathcal{B} = Id + \mathcal{W}' \tag{3.8}$$

and from Lemma 3.3 we derive for $j = 1, 2$

$$\gamma(\mathcal{K}_1\psi|_{\Omega_j}) = \frac{1}{2}(\mathcal{W} + (-1)^j Id)\psi \ , \ \ \delta(\mathcal{K}_0\chi|_{\Omega_j}) = \frac{1}{2}(\mathcal{W}' - (-1)^j Id)\chi\,. \tag{3.9}$$

Therefore we call $\mathcal{W}$ the double layer potential operator of the bi–Laplacian on Γ. The corresponding single layer potential operator on Γ satisfies a Gårding inequality.

Lemma 3.4. *The operator $\mathcal{A}$ is strongly elliptic, i.e. there exist a compact operator $\mathcal{T} : (V(\Gamma))' \to V(\Gamma)$ and a positive constant c such that*

$$|[\chi, (\mathcal{A} + \mathcal{T})\chi]| \geq c\,\|\chi\|^2_{(V(\Gamma))'}\,, \quad \forall\, \chi \in (V(\Gamma))'\,.$$

Proof. For $\chi \in (V(\Gamma))'$ and $u = -\mathcal{K}_0\chi$ we have the relations

$$\gamma u|_{\Omega_1} = \gamma u|_{\Omega_2} = -\frac{1}{2}\mathcal{A}\chi\,, \quad \{\delta u\} = \chi\,.$$

We choose $\varphi \in C_0^\infty(\mathbb{R}^2)$ with $\varphi \equiv 1$ on a neighbourhood of Ω_1 and set $u_1 = u|_{\Omega_1}$, $u_2 = \varphi u|_{\Omega_2}$. Then

$$\frac{1}{2}\,[\chi, \mathcal{A}\chi] = [\delta u_1, \gamma u_1] - [\delta u_2, \gamma u_2] = \int\limits_{\Omega_1} |\Delta u_1|^2 dx + \int\limits_{\Omega_2} |\Delta u_2|^2 dx - \int\limits_{\Omega_2} u_2\,\Delta^2 u_2\,dx\,.$$

Note that for any function $u \in H^2(\Omega_1)$ there holds

$$\int\limits_{\Omega_1} \Big(\frac{\partial^2 u}{\partial x_1 \partial x_2}\Big)^2 dx = \int\limits_{\Omega_1} \frac{\partial^2 u}{\partial x_1^2}\,\frac{\partial^2 u}{\partial x_2^2}\,dx + \Big\langle \frac{\partial u}{\partial x_2}, \partial_s\,\frac{\partial u}{\partial x_1}\Big\rangle_\Gamma\,,$$

which follows from Green's formula applied to sufficiently smooth functions, Lemma 2.1 and density arguments. Since $\gamma u_1 = \gamma u_2$ we conclude that

$$\frac{1}{2}\,[\chi, \mathcal{A}\chi] = |u_1|^2_{H^2(\Omega_1)} + |u_2|^2_{H^2(\Omega_2)} - \int\limits_{\Omega_2} u_2\,\Delta^2 u_2\,dx\,,$$

where $|\cdot|_{H^2(\Omega_j)}$ denotes the usual seminorm. Thus

$$\begin{aligned}\|\chi\|^2_{(V(\Gamma))'} &= \|\delta u_1 - \delta u_2\|^2_{(V(\Gamma))'} \leq c\Big(\|u_1\|^2_{H^2(\Omega_1)} + \|u_2\|^2_{H^2(\Omega_2)} + \|\Delta^2 u_2\|^2_{L^2(\Omega_2)}\Big) \\ &\leq \frac{c}{2}\,[\chi, \mathcal{A}\chi] + c\Big(\|u_1\|^2_{L^2(\Omega_1)} + \|u_2\|^2_{L^2(\Omega_2)} + \|\Delta^2 u_2\|^2_{L^2(\Omega_2)} + \int\limits_{\Omega_2} u_2\,\Delta^2 u_2\,dx\Big)\,.\end{aligned}$$

Since $\Delta^2 u_2$ has a compact support in Ω_2 and is C^∞, the term in the brackets is generated by a compact bilinear form of χ. ■

Corollary 3.2. *The operator*

$$\mathcal{A} : (V(\Gamma))' \to V(\Gamma)$$

is Fredholm with index zero. If $\mathcal{A}\chi \in V(\Gamma)$ *then* $\chi \in (V(\Gamma))'$.

4. Calderon projections

Now we are in the position to define the Calderon projections which map onto the Cauchy data of functions biharmonic in Ω_1 or Ω_2. Here we follow a method developed in [10] for second order equations.

We define the linear spaces

$$L_j := \{u(x) = \mathcal{K}_0\chi(x) - \mathcal{K}_1\psi(x) : (\psi, \chi) \in V(\Gamma) \times (V(\Gamma))',\, x \in \Omega_j\}\ ,$$

in which solutions of the biharmonic equation are sought. From the Lemmas 3.1 and 3.2 we conclude that L_1 is the set of functions $u \in H^2(\Omega_1)$ satisfying $\Delta^2 u = 0$. Moreover, for $u \in L_1$ the representation formula

$$\mathcal{K}_0\delta u(x) - \mathcal{K}_1\gamma u(x) = \begin{cases} u(x) & ,\ x \in \Omega_1\ , \\ 0 & ,\ x \in \Omega_2\ , \end{cases} \tag{4.1}$$

holds.

The space L_2 consists of $u \in H^2_{loc}(\Omega_2)$ providing $\Delta^2 u = 0$ and a special behaviour at infinity, which we refer as *radiation condition*. The asymptotics of functions belonging to L_2 can be described as follows:

Using the functions

$$g_1(x,y) = 1\ , \qquad g_2(x,y) = \frac{x\cdot y}{|x|}\ ,$$
$$g_3(x,y) = |y|^2\ , \qquad g_4(x,y) = \frac{|y|^2}{2} + \frac{(x\cdot y)^2}{|x|^2}\ ,$$

(here $x\cdot y$ denotes the inner product of vectors $x, y \in \mathbb{R}^2$ and $|y|^2 = y\cdot y$), we introduce

$$\begin{aligned} I_j\chi(x) &= [\chi, \gamma g_j(x,\cdot)] \quad ,\ \chi \in (V(\Gamma))'\ , \quad j = 1(1)4\ , \\ I_5\psi &= [\delta g_3(x,\cdot), \psi] \quad ,\ \psi \in V(\Gamma)\ . \end{aligned} \tag{4.2}$$

Note that I_1, I_3 and I_5 are constants while I_2 and I_4 depend on the direction of x.

Lemma 4.1. *For given* $(\psi, \chi) \in V(\Gamma) \times (V(\Gamma))'$ *the function*

$$u(x) = \mathcal{K}_1\psi(x) - \mathcal{K}_0\chi(x)$$

behaves for large $|x| = R$ *as*

$$\begin{aligned} u(x) = &-\frac{1}{8\pi}\Big(I_1\chi\, R^2 \ln R - I_2\chi(x)(2R\ln R + R) + (I_3\chi - I_5\psi)\ln R + I_4\chi(x) - I_5\psi\Big) \\ &+ O(R^{-1})\,. \end{aligned} \tag{4.3}$$

This expansion was proved in [4] for the case of ψ and χ having continuous components, such that from Lemmas 2.2 and 2.7 the assertion follows immediately.

A representation formula similar to (4.1) holds also for functions $u \in L_2$.

Lemma 4.2. *For* $u \in L_2$ *with Cauchy data* $(\gamma u, \delta u)$ *there holds*

$$\mathcal{K}_1\gamma u(x) - \mathcal{K}_0\delta u(x) = \begin{cases} u(x) & ,\ x \in \Omega_2\ , \\ 0 & ,\ x \in \Omega_1\ . \end{cases} \tag{4.4}$$

Proof . We enclose Ω_1 by a ball B_R with radius $R > |x|$. Then the representation formula (4.1) is valid for the bounded domain $\Omega_1 \cap B_R$ yielding

$$
\begin{aligned}
&u(x) = \mathcal{K}_1 \gamma u(x) - \mathcal{K}_0 \delta u(x) \\
&+ \int\limits_{S_R} \Big(u\, \partial_{n_z} \Delta G(x,z) - \Delta G(x,z)\, \partial_n u + \Delta u\, \partial_{n_z} G(x,z) - G(x,z)\, \partial_n \Delta u \Big)\, ds_z \,.
\end{aligned}
$$

Using the asymptotics (4.3) of $u(z)$ as $R = |z| \to \infty$ and the asymptotics of the fundamental solution given in [4]

$$
G(x,z) = \frac{1}{8\pi}\Big(R^2 \ln R - (x\cdot n_z)(2R\ln R + R) + |x|^2 \ln R + \frac{|x|^2}{2} + (x\cdot n_z)^2 \Big) + O(R^{-1})\,,
$$

one obtains with the help of a computer algebra system that the integrand permits the expansion

$$
\begin{aligned}
&\frac{1}{64\pi^2}\Big(\big((x\cdot n_z)\, I_1 \delta u - I_2 \delta u(z)\big)\big(3(\ln R - 1) - 2(\ln R)^2\big) \\
&\qquad - \frac{2}{R}\big((2(x\cdot n_z)^2 - |x|^2)\, I_1 \delta u + 2 I_3 \delta u - 2 I_4 \delta u(z)\big)\Big) + O(R^{-2})\,.
\end{aligned}
$$

Obviously

$$
\int\limits_{S_R} (x\cdot n_z)\, ds_z = \int\limits_{S_R} I_2 \delta u(z)\, ds_z = 0\,,
$$

such that the integral of the first term in the brackets vanishes. Further, denote by θ_z the angle between x and the integration point z. Then

$$
2(x\cdot n_z)^2 - |x|^2 = |x|^2(2\cos^2\theta_z - 1) = |x|^2 \cos 2\theta_z\,,
$$

implying

$$
\int\limits_{S_R} \big(2(x\cdot n_z)^2 - |x|^2\big)\, I_1 \delta u\, ds_z = 0\,.
$$

Finally, we have

$$
I_4 \delta u(z) - I_3 \delta u = [\delta u, \gamma h(z,\cdot)]
$$

with the function

$$
h(z,y) = \frac{(z\cdot y)^2}{|z|^2} - \frac{|y|^2}{2} = \frac{|y|^2}{2} \cos 2\theta_z\,,
$$

where now θ_z is the angle between y and z. Denoting by α the angle between y and n_y we get

$$
\begin{aligned}
\partial_{n_y} h(z,y) &= 2(y\cdot n_z)(n_y\cdot n_z) - (y\cdot n_y) \\
&= |y|(2\cos\theta_z \cos(\theta_z - \alpha) - \cos\alpha) = |y|\cos(2\theta_z - \alpha)\,.
\end{aligned}
$$

Hence

$$
\int\limits_{S_R} \big(I_3 \delta u - I_4 \delta u(z)\big)\, ds_z = 0\,,
$$

such that

$$
\int\limits_{S_R} \Big(u\, \partial_{n_z} \Delta G(x,z) - \Delta G(x,z)\, \partial_n u + \Delta u\, \partial_{n_z} G(x,z) - G(x,z)\, \partial_n \Delta u \Big)\, ds_z = O(R^{-1})\,. \ \blacksquare
$$

Now we introduce the linear operator

$$
\mathfrak{A} := \begin{pmatrix} -\mathcal{W} & \mathcal{A} \\ \mathcal{O} & \mathcal{W}' \end{pmatrix} : \begin{array}{c} V(\Gamma) \\ \times \\ (V(\Gamma))' \end{array} \to \begin{array}{c} V(\Gamma) \\ \times \\ (V(\Gamma))' \end{array}\,,
$$

where $\mathcal{O}$ denotes the zero mapping, and define

$$\mathfrak{P}_j := \frac{1}{2}(Id - (-1)^j \mathfrak{A}) , \quad j = 1, 2 . \tag{4.5}$$

Theorem 4.1. *The operators $\mathfrak{P}_j$ are bounded projections in $V(\Gamma) \times (V(\Gamma))'$ mapping onto the set of Cauchy data $(\gamma u, \delta u)$ of functions $u \in L_j$.*

Proof . The boundedness of $\mathfrak{P}_j$ follows from Lemma 3.2. Further, for any $(\psi, \chi) \in V(\Gamma) \times (V(\Gamma))'$ we have

$$u = (-1)^j (\mathcal{K}_1 \psi - \mathcal{K}_0 \chi) \in L_j$$

and by Lemma 3.2 and (3.9)

$$\begin{pmatrix} \gamma u \\ \delta u \end{pmatrix} = (-1)^j \begin{pmatrix} \gamma(\mathcal{K}_1 \psi|_{\Omega_j}) - \gamma(\mathcal{K}_0 \chi|_{\Omega_j}) \\ \delta(\mathcal{K}_1 \psi|_{\Omega_j}) - \delta(\mathcal{K}_0 \chi|_{\Omega_j}) \end{pmatrix} = (-1)^j \begin{pmatrix} \frac{1}{2}(\mathcal{W} + (-1)^j Id)\psi - \frac{1}{2}\mathcal{A}\chi \\ -\frac{1}{2}(\mathcal{W}' - (-1)^j Id)\chi \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} Id + (-1)^j \mathcal{W} & -(-1)^j \mathcal{A} \\ \mathcal{O} & Id - (-1)^j \mathcal{W}' \end{pmatrix} \begin{pmatrix} \psi \\ \chi \end{pmatrix} = \frac{1}{2}(Id - (-1)^j \mathfrak{A}) \begin{pmatrix} \psi \\ \chi \end{pmatrix} = \mathfrak{P}_j \begin{pmatrix} \psi \\ \chi \end{pmatrix} .$$

Let now $u \in L_j$. Then the representation formulas (4.1) and (4.4) yield

$$u(x) = (-1)^j (\mathcal{K}_1 \gamma u(x) - \mathcal{K}_0 \delta u(x)) , \quad x \in \Omega_j ,$$

after applying the jump relations of Lemma 3.3 and (3.9) we obtain

$$\begin{pmatrix} \gamma u \\ \delta u \end{pmatrix} = \mathfrak{P}_j \begin{pmatrix} \gamma u \\ \delta u \end{pmatrix} ,$$

showing that the mappings $\mathfrak{P}_j$ are projections and that the Cauchy data of all functions from L_j belong to the image of $\mathfrak{P}_j$. ■

Since the Calderon projections corresponding to the interior and the exterior problem are conjugate

$$\mathfrak{P}_1 + \mathfrak{P}_2 = Id ,$$

the space $V(\Gamma) \times (V(\Gamma))'$ can be decomposed as the direct sum of closed subspaces

$$V(\Gamma) \times (V(\Gamma))' = \{(\gamma u, \delta u) : u \in L_1\} \dot{+} \{(\gamma u, \delta u) : u \in L_2\} .$$

Further, since $\mathfrak{P}_j^2 = \mathfrak{P}_j$ we get

Corollary 4.1.

$$\frac{1}{4}(Id \pm \mathcal{W})^2 = \frac{1}{2}(Id \pm \mathcal{W}) \quad , \quad \mathcal{W}\mathcal{A} = \mathcal{A}\mathcal{W}' \tag{4.6}$$

5. Steklov-Poincaré operators

In this section we derive equations with the strongly elliptic single layer potential operator $\mathcal{A}$ for the solution of the interior and of the exterior Dirichlet problem

$$\begin{aligned} &\Delta^2 u = 0 \quad \text{in } \Omega_j , \qquad \gamma u = \psi \in V(\Gamma) , \\ &\text{if } j = 2 \text{ then } u \text{ satisfies the radiation condition (4.3)} , \end{aligned} \tag{5.1}$$

and study the corresponding solution operators.

From Theorem 4.1 we know that any function $u \in L_j$ satisfies the relation

$$(Id - \mathfrak{P}_j) \begin{pmatrix} \gamma u \\ \delta u \end{pmatrix} = 0 , \tag{5.2}$$

the first line of this system yields in particular the equality

$$(Id - (-1)^j \mathcal{W})\gamma u + (-1)^j \mathcal{A}\delta u = 0 .$$

Hence, if we consider the Dirichlet problem (5.1) then for given $\gamma u = \psi$ the unknown $\chi = \delta u$ has to solve the equation

$$\mathcal{A}\chi = (\mathcal{W} - (-1)^j Id)\, \psi\,. \tag{5.3}$$

In order to study the solvability of these equations we make the assumption

A1: The exterior homogeneous Dirichlet problem (5.1), i.e. $\psi = 0$, has only the trivial solution.

Theorem 5.1. *Suppose* **A1**. *The equations* (5.3) *are uniquely solvable for any* $\psi \in V(\Gamma)$ *and the weak solution* $u \in L_j$ *of the corresponding Dirichlet problem* (5.1) *is given by*

$$u(x) = (-1)^j(\mathcal{K}_1\psi(x) - \mathcal{K}_0\chi(x))\,, \quad x \in \Omega_j\,.$$

Proof. The unique solvability of the interior Dirichlet problem (Lemma 2.6) and the jump relations for the operator $\mathcal{K}_0$ (Lemma 3.3) imply that the equation

$$\mathcal{A}\chi = 0$$

has a nontrivial solution if and only if our assumption does not hold. Since by Corollary 3.2 $\mathcal{A}$ is Fredholm with index zero we derive that $\mathcal{A} : (V(\Gamma))' \to V(\Gamma)$ is bijective. ■

Remark 5.1. For a smooth boundary Γ and the interior Dirichlet problem this result follows from the general theory of boundary integral operators developed in [6] and [12]. It was formulated in [9].

Now we analyse the solution operators of the equations (5.3)

$$\mathcal{T}_j := \mathcal{A}^{-1}(\mathcal{W} - (-1)^j Id)\,:\, V(\Gamma) \to (V(\Gamma))' \tag{5.4}$$

which exist under assumption **A1** and map the Dirichlet data γu of a biharmonic function $u \in L_j$ to its Neumann data δu. The mappings $\mathcal{T}_j$ are the *Steklov-Poincaré operators* of the biharmonic equation.

Let us define the operators

$$\mathcal{P}_j := \frac{1}{2}(Id - (-1)^j \mathcal{W}) : V(\Gamma) \to V(\Gamma)\,, \tag{5.5}$$

which are bounded projections by Corollary 4.1. In the following Lemma we prove that these operators coincide with the well known Calderon projections for the Laplace equation, but corresponding to the fundamental solution

$$g(x,y) := -\Delta_x G(x,y) = -\frac{1}{2\pi}(\ln|x-y| + 1) = -\frac{1}{2\pi}\ln(e|x-y|)\,.$$

Lemma 5.1. *It holds*

$$V(\Gamma) = V_1 \dotplus V_2$$

with the closed subspaces

$$\begin{aligned} V_1 := \operatorname{im} \mathcal{P}_1 = \{\gamma u \,:\, u \in H^2_{loc}(\Omega_2)\,,\ \Delta u = 0\,,\\ u(x) = a(\ln|x| + 1) + O(|x|^{-1}) \ \textit{for some}\ a \in \mathbb{R} \ \textit{as}\ |x| \to \infty\}\,, \end{aligned}$$

$$V_2 := \operatorname{im} \mathcal{P}_2 = \{\gamma u \,:\, u \in H^2(\Omega_1)\,,\ \Delta u = 0\}\,.$$

Proof. We define the boundary integral operators for $x \in \Gamma$

$$S\varphi(x) := 2\int_\Gamma g(x,y)\varphi(y)ds_y\,, \quad D\varphi(x) := 2\int_\Gamma \partial_{n_y} g(x,y)\varphi(y)ds_y\,,$$

$$D'\varphi(x) := 2\,\partial_{n_x}\int_\Gamma g(x,y)\varphi(y)ds_y\,,\ H\varphi(x) := -2\,\partial_{n_x}\int_\Gamma \partial_{n_y} g(x,y)\varphi(y)ds_y\,. \tag{5.6}$$

It is clear that D' is the adjoint of the operator D with respect to the L^2-inner product on Γ. Using formula (3.6) and the jump relations of harmonic potentials it is easy to see that for $\psi = \binom{v_1}{v_2} \in V(\Gamma)$ it holds

$$\mathcal{W}\psi = (Id + \mathcal{C})\psi = \begin{pmatrix} D & -S \\ -H & -D' \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} . \tag{5.7}$$

From the results in [10] it is evident that the mappings

$$\frac{1}{2}(Id + (-1)^k \mathcal{W})$$

are bounded in $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ and project onto the boundary values of weak solutions of the Laplace equation in Ω_k, behaving for $k = 2$ at infinity as

$$u(x) = a \ln(e|x|) + O(|x|^{-1}) = a(\ln|x| + 1) + O(|x|^{-1}) . \tag{5.8}$$

By Lemma 3.2 the restrictions of these projections are bounded in $V(\Gamma)$. ■

Note that due to the definition (5.5) the mappings $\mathcal{P}_j$ appearing on the right–hand side of the boundary integral equation (5.3) for the interior ($j = 1$) and exterior ($j = 2$) Dirichlet problem project onto the traces of functions harmonic on the opposite domain.

The dual space $(V(\Gamma))'$ is the direct sum of the corresponding polar sets

$$(V(\Gamma))' = V_1^{\perp} \dot{+} V_2^{\perp} ,$$

which in view of

$$V_j^{\perp} = (\operatorname{im} \mathcal{P}_j)^{\perp} = \ker \mathcal{P}_j' = \operatorname{im}(Id - \mathcal{P}_j') \tag{5.9}$$

coincide with the image of the adjoint of the conjugate projection. The commutative relation (4.6) implies that

$$\mathcal{A}\mathcal{P}_j' = \mathcal{P}_j \mathcal{A} \mathcal{P}_j' = \mathcal{P}_j \mathcal{A} ,$$

yielding the equality

$$\mathcal{A} = \mathcal{P}_1 \mathcal{A} \mathcal{P}_1' + \mathcal{P}_2 \mathcal{A} \mathcal{P}_2' .$$

Using (5.9) and Theorem 5.1 we derive

Lemma 5.2. *The operator $\mathcal{A}$ is the direct sum of the mappings*

$$\mathcal{A} : V_1^{\perp} \to V_2 \quad \text{and} \quad \mathcal{A} : V_2^{\perp} \to V_1 ,$$

which are bijective if the assumption **A1** *is satisfied.*

Now we show that $\mathcal{A}$ is a positive definite operator on a subspace of $(V(\Gamma))'$. Let us denote by $\mathbb{P}_1$ the space of linear functions on $\mathbb{R}^2$ and set $l(\Gamma) := \gamma(\mathbb{P}_1)$.

Lemma 5.3. *There exists a constant $c > 0$ such that for any $\chi \in l(\Gamma)^{\perp}$ there holds*

$$[\chi, \mathcal{A}\chi] \geq c\, \|\chi\|^2_{(V(\Gamma))'} .$$

Proof. We set $u = -\mathcal{K}_0\chi$, $u_1 = u|_{\Omega_1}$ and $u_2 = u|_{\Omega_2}$. For any ball B_R enclosing Ω_1 the first Green formula yields

$$\begin{aligned} \frac{1}{2}[\chi, \mathcal{A}\chi] &= [\delta u_1, \gamma u_1] - [\delta u_2, \gamma u_2] \\ &= \int_{\Omega_1} |\Delta u_1|^2 dx + \int_{\Omega_2 \cap B_R} |\Delta u_2|^2 dx - \int_{S_R} (\Delta u_2\, \partial_n u_2 - u_2\, \partial_n \Delta u_2)\, ds . \end{aligned}$$

Because of $\chi \in l(\Gamma)^{\perp}$ and the definition (4.2) it is clear that $I_1\chi = I_2\chi(x) = 0$ leading to

$$u_2(x) = -\frac{1}{8\pi}\big(I_3\chi \ln R + I_4\chi(x)\big) + O(R^{-1}) \quad \text{for } |x| = R .$$

Hence, $\Delta u_2 \in L^2(\Omega_2)$ and the integral over S_R converges to zero as $R \to \infty$ such that

$$[\chi, \mathcal{A}\chi] = 2\Big(\int\limits_{\Omega_1} |\Delta u_1|^2 dx + \int\limits_{\Omega_2} |\Delta u_2|^2 dx \Big) > 0$$

for $\chi \neq 0$. Since $\mathcal{A}$ is symmetric and strongly elliptic the last inequality implies that $\mathcal{A}$ is even positive definite on $l(\Gamma)^{\perp}$. ■

Remark 5.2. Since $l(\Gamma)^{\perp}$ can be identified with the dual of the factor space $V(\Gamma)/l(\Gamma)$ it is evident that

$$[\chi, \mathcal{A}\chi] \geq c\, \|\chi\|^2_{(V(\Gamma)/l(\Gamma))'} \quad \forall\, \chi \in (V(\Gamma)/l(\Gamma))' \, .$$

This was used by Bourlard in [2] to prove the existence of the solution $u \in L_1$ of (5.1) in the form

$$u(x) = \lfloor \mathcal{K}_0\chi \rfloor(x) + p_1(x) \quad , \; x \in \Omega_1 \, ,$$

where $\chi \in (V(\Gamma)/l(\Gamma))'$ solves

$$[\varphi, \mathcal{A}\chi] = 2\,[\varphi, \psi] \quad , \; \forall\, \varphi \in (V(\Gamma)/l(\Gamma))' \, ,$$

$\lfloor \mathcal{K}_0\chi \rfloor$ is an element of the corresponding factor class in $L_1/\mathbb{P}_1$ and $p_1 \in \mathbb{P}_1$ is the linear function satisfying

$$\gamma p_1 = \psi - \gamma \lfloor \mathcal{K}_0\chi \rfloor \, .$$

Now we come to some consequences of the previous results.

Corollary 5.1. *The restriction of* $\mathcal{A}$ *on* $V_2^{\perp} \subset (V(\Gamma))'$ *is a symmetric and positive definite operator between the dual spaces*

$$\mathcal{A} : V_2^{\perp} = \operatorname{im} \mathcal{P}_1' \to V_1 = \operatorname{im} \mathcal{P}_1 \, .$$

If the assumption **A1** *is violated then* $\ker \mathcal{A} \subset V_1^{\perp}$ *and* $\ker \mathcal{A} \cap l(\Gamma)^{\perp} = \emptyset$*.*

Corollary 5.2. $\chi \in (V(\Gamma))'$ *coincides with the Neumann data* δu *of a function* $u \in L_j$ *if and only if* $[\chi, \gamma v] = 0$ *for any harmonic function* $v \in H^2_{loc}(\Omega_j)$ *satisfying, if* $j = 2$*, additionally the radiation condition* (5.8).

Corollary 5.3. *If* $\chi \in V_j^{\perp}$ *then the function* $\mathcal{K}_0\chi \in H^2_{loc}(\Omega_j)$ *is harmonic in* Ω_j *and satisfies, in the case* $j = 2$*, the radiation condition* (5.8).

Proof . Corollary 5.2 states that for any $\chi \in V_j^{\perp}$ there exist $u \in L_k$, $k = 3 - j$, such that $\chi = \delta u$. The representation formulas (4.1) and (4.4) imply that

$$\mathcal{K}_0\chi(x) = \begin{cases} \mathcal{K}_1\gamma u(x) - (-1)^k u(x) & , \; x \in \Omega_k \, , \\ \mathcal{K}_1\gamma u(x) & , \; x \in \Omega_j \, . \end{cases} \quad ■$$

Now we are in the position to formulate some properties of the Steklov-Poincaré operators. By (5.4) and (5.5) we get

$$\mathcal{T}_j = 2 \cdot (-1)^{j+1}\, \mathcal{A}^{-1}\, \mathcal{P}_j = 2 \cdot (-1)^{j+1}\, \mathcal{P}_j'\, \mathcal{A}^{-1}\, \mathcal{P}_j$$

such that the following assertions hold.

Theorem 5.2. *The Steklov-Poincaré operator* $\mathcal{T}_1$ *which maps the Dirichlet data* γu *of a function* $u \in H^2(\Omega_1)$ *biharmonic on the bounded domain* Ω_1 *with piecewise smooth boundary* Γ *to its Neumann data* δu *is continuous from* $V(\Gamma)$ *into* $(V(\Gamma))'$*, symmetric with respect to the duality* (2.1) *and there exists* $c > 0$ *such that*

$$[\mathcal{T}_1\psi, \psi] \geq c\, \|\mathcal{P}_1\psi\|_{V(\Gamma)} \quad , \; \forall\, \psi \in V(\Gamma) \, .$$

Moreover, the image $\operatorname{im} \mathcal{T}_1 \subset (V(\Gamma))'$ *is the closed subspace of elements which are orthogonal to the traces* γv *of all harmonic functions* $v \in H^2(\Omega_1)$*.*

Theorem 5.3. *Suppose* **A1**. *Then the Steklov-Poincaré operator $\mathcal{T}_2$ which maps the Dirichlet data γu of a function $u \in L_2$ to its Neumann data δu is continuous from $V(\Gamma)$ into $(V(\Gamma))'$, symmetric with respect to the duality (2.1) and there exists $c > 0$ such that*

$$-[\mathcal{T}_2\psi, \psi] \geq c\, \|\mathcal{P}_2\psi\|_{V(\Gamma)} \quad , \; \forall\, \psi \in \mathcal{A}(l\,(\Gamma)^{\perp})\,.$$

The image im $\mathcal{T}_2 \subset (V(\Gamma))'$ *is the closed subspace of elements which are orthogonal to the traces γv of all harmonic functions $v \in H^2_{loc}(\Omega_2)$ satisfying the radiation condition (5.8).*

Remark 5.3. The previous results confirm the well-known fact that the Neumann problem

$$\Delta^2 u = 0 \quad \text{in } \Omega_j\,, \qquad \delta u = \chi \in (V(\Gamma))'$$

is not elliptic. A variational approach to boundary conditions different from the Dirichlet one is based on the bilinear form

$$\int\limits_{\Omega_1} \left(\Delta u\, \Delta v + (1-\sigma)\Big(2 \frac{\partial^2 u}{\partial x_1 \partial x_2} \frac{\partial^2 v}{\partial x_1 \partial x_2} - \frac{\partial^2 u}{\partial x_1^2} \frac{\partial^2 v}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_2^2} \frac{\partial^2 v}{\partial x_1^2} \Big) \right) dx \;,\; 0 < \sigma < 1\,, \tag{5.10}$$

which is closely connected with the plate equation. A detailed analysis of certain indirect integral equation methods on smooth boundaries for these problems is contained in the book [3] of Chen and Zhou. The case of a nonsmooth curve Γ has not been analysed in the literature, up to now. It is possible to modify our methods accordingly to the form (5.10) such that direct boundary integral equations for plate problems on domains with corners can be derived and analysed.

6. Boundary integral equations for Dirichlet problems

In this section we derive systems of integral equations for the interior and exterior Dirichlet problem. We consider the existence and uniqueness of solutions and discuss the assumption **A1**.

First we consider the concrete form of the mappings $\mathcal{A}$ and $\mathcal{P}_j$ which are 2×2 matrices of integral operators. In view of (2.1) and (3.4) the action of the operator $\mathcal{A}$ can be written as

$$\mathcal{A}\chi = \begin{pmatrix} A & -B \\ B' & C \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad , \quad \chi = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in (V(\Gamma))' \tag{6.1}$$

with the integral operators

$$A\varphi(x) := -2 \int\limits_{\Gamma} G(x,y)\, \varphi(y) ds_y\,, \quad B\varphi(x) := -2 \int\limits_{\Gamma} \partial_{n_y} G(x,y)\, \varphi(y) ds_y\,,$$

$$B'\varphi(x) := -2\, \partial_{n_x} \int\limits_{\Gamma} G(x,y)\, \varphi(y) ds_y\,, \quad C\varphi(x) := 2\, \partial_{n_x} \int\limits_{\Gamma} \partial_{n_y} G(x,y)\, \varphi(y) ds_y\,.$$

By the duality (2.1) and (5.7) we have

$$\mathcal{W}' = \begin{pmatrix} D' & S \\ H & -D \end{pmatrix},$$

hence the commutative relation (4.6) leads to the equalities

$$\begin{aligned} A\,D' - B\,S = D\,A - S\,B' \quad &, \; A\,H + B\,D = -D\,B - S\,C\,, \\ B'\,D' + C\,S = -H\,A - D'\,B'\,&, \; B'\,H - C\,D = H\,B - D'C\,. \end{aligned} \tag{6.2}$$

Accordingly to (5.5) the projections $\mathcal{P}_j$ have the form

$$\mathcal{P}_j = \frac{1}{2} \begin{pmatrix} I-(-1)^j D & (-1)^j S \\ (-1)^j H & I+(-1)^j D' \end{pmatrix}, \quad \mathcal{P}_j' = \frac{1}{2} \begin{pmatrix} I-(-1)^j D' & -(-1)^j H \\ -(-1)^j S & I+(-1)^j D \end{pmatrix},$$

implying the relations

$$\begin{aligned} D^2 + S\,H = D'^2 + H\,S = I\ , \\ D\,S = S\,D\ ,\ \ H\,D = D'\,H\ . \end{aligned} \tag{6.3}$$

For the following we mention some other properties of the boundary integral operators (5.6) for the Laplace equation.

It is well known that there exists a unique ρ, the Robin potential, which fulfills $\langle \rho, 1\rangle_\Gamma = 1$ and belongs additionally to $H^{-1/2}(\Gamma)$ such that the logarithmic potential

$$\int_\Gamma \rho(y) \ln|x-y|\, ds_y$$

is constant (say $= \nu$) on Γ. The positive number

$$cap\ \Gamma = e^{\nu}$$

is called the *logarithmic capacity* of Γ. We introduce the assumption

A2: The curve Γ is such that $cap\ \Gamma \neq e^{-1}$.

Remark that **A2** means that the exterior Dirichlet problem for the Laplace equation

$$\Delta v = 0 \quad \text{in } \Omega_2\ , \quad v|_\Gamma = 0\ , \quad v \text{ satisfies (5.8)}\ ,$$

has in $H^1_{loc}(\Omega_2)$ only the trivial solution $v = 0$.

Evidently, if **A2** holds then the operator S has a trivial kernel. Moreover, S maps $H^{-1/2}(\Gamma)$ isomorphically onto $H^{1/2}(\Gamma)$ and the subspaces $V_j \subset V(\Gamma)$ can be characterized by the relation

$$\psi = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in V_j \quad \Longleftrightarrow \quad v_2 = S^{-1}\,(D + (-1)^j I)\,v_1\ . \tag{6.4}$$

Turning to the duals we obtain the characterization of $V_j^\perp \subset (V(\Gamma))'$

$$\chi = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in V_j^\perp \quad \Longleftrightarrow \quad v_1 = S^{-1}\,(D + (-1)^j I)\,v_2\ , \tag{6.5}$$

which means of course $\langle v_1, \varphi|_\Gamma\rangle_\Gamma = \langle v_2, S^{-1}\,(D + (-1)^j I)\varphi|_\Gamma\rangle_\Gamma$ for all $\varphi \in C_0^\infty(\mathbb{R}^2)$.

Concerning the double layer potential D we note that the kernel of the operator $I - D$ is trivial, whereas the operators $I + D$ and $I + D'$ have onedimensional kernels spanned by the constant function on Γ and by the Robin potential ρ, respectively.

The mentioned properties can be easily deduced from known results about harmonic potentials corresponding to the fundamental solution $-\frac{1}{2\pi}\ln|x-y|$, from Corollary 4.1 and the fact that the projections $\mathcal{P}_j$ are bounded in $V(\Gamma)$.

In Section 5 we have studied already the equations for solving the interior ($j = 1$) and exterior ($j = 2$) Dirichlet problem

$$\mathcal{A}\chi = 2 \cdot (-1)^{j+1}\,\mathcal{P}_j\,\psi\ ,$$

which can be written as the system

$$\begin{aligned} A v_1 - B v_2 &= (D - (-1)^j I) f_1 - S f_2 \\ B' v_1 + C v_2 &= -H f_1 - (D' + (-1)^j I) f_2\ , \end{aligned} \tag{6.6}$$

where $\binom{f_1}{f_2} = \psi \in V(\Gamma)$ are the given Dirichlet data and $\binom{v_1}{v_2} = \delta u = \chi \in (V(\Gamma))'$ are the unknowns. From the results of Section 5 follows that under the assumption **A1** the unique solution χ belongs to the closed subspace

$$\chi \in \operatorname{im} \mathcal{P}_j' = \ker(I - \mathcal{P}_j')\ .$$

Consequently, if **A1** is satisfied then for $j = 1, 2$ the solution $\binom{v_1}{v_2} \in (V(\Gamma))'$ of (6.6) solves in view of (6.5) the corresponding systems of boundary integral equations

$$\begin{aligned} Av_1 - Bv_2 &= (D - (-1)^j I)f_1 - Sf_2 \\ Sv_1 - (D - (-1)^j I)v_2 &= 0 \;. \end{aligned} \tag{6.7}$$

and

$$\begin{aligned} B'v_1 + Cv_2 &= -Hf_1 - (D' + (-1)^j I)f_2 \\ Sv_1 - (D - (-1)^j I)v_2 &= 0 \;. \end{aligned} \tag{6.8}$$

To consider the opposite direction we assume **A2** and use the equality

$$\begin{aligned} &B'\,S^{-1}\,(D - (-1)^j I) + C \\ &\quad = S^{-1}\,(D + (-1)^j I)(A\,S^{-1}\,(D - (-1)^j I) - B)\;, \end{aligned} \tag{6.9}$$

which follows immediately from (6.2) and (6.3). Indeed,

$$\begin{aligned} &B'\,S^{-1}\,(D - (-1)^j I) + C = (B'\,(D' - (-1)^j I) + C\,S)\,S^{-1} \\ &= (-H\,A - D'\,B - (-1)^j B')\,S^{-1} = S^{-1}\,((D^2 - I)\,A - (D + (-1)^j I)\,S\,B')\,S^{-1} \\ &= S^{-1}\,(D + (-1)^j I)(-(-1)^j A + A\,D' - B\,S)\,S^{-1} \\ &= S^{-1}\,(D + (-1)^j I)(A\,S^{-1}\,(D - (-1)^j I) - B)\;. \end{aligned}$$

Furthermore, (6.4) shows that

$$-Hf_1 - (D' + (-1)^j I)f_2 = S^{-1}\,(D + (-1)^j I)((D - (-1)^j I)f_1 - Sf_2)\;.$$

Comparing with (6.9) we see that the second equation of (6.6) is a consequence of the first equation of this system and (6.5), i.e. the second equation of (6.7). Using the fact that $I - D$ is invertible we obtain for the case of the interior problem ($j = 1$) that the first equation of (6.6) holds if the second equation of this system and (6.5) are satisfied.

Consequently, the assumption **A2** implies that any solution of the system (6.7) solve (6.6), too. Moreover, in the case $j = 1$, i.e. the interior problem, any solution of the system (6.8) solves (6.6). Thus we derive

Theorem 6.1. *Suppose that Γ satisfies the assumption* **A2**. *For any $\psi = \binom{f_1}{f_2} \in V(\Gamma)$ the systems of boundary integral equations*

$$\begin{aligned} Av_1 - Bv_2 &= (D + I)f_1 - Sf_2 \\ Sv_1 - (D + I)v_2 &= 0 \end{aligned} \tag{6.10}$$

and

$$\begin{aligned} B'v_1 + Cv_2 &= -Hf_1 - (D' - I)f_2 \\ Sv_1 - (D + I)v_2 &= 0 \end{aligned} \tag{6.11}$$

are uniquely solvable. The solution $\chi = \binom{v_1}{v_2} \in (V(\Gamma))'$ coincides with the Neumann data δu of $u \in H^2(\Omega_1)$ solving the interior Dirichlet problem

$$\Delta^2 u = 0 \quad in\ \Omega_1\,, \qquad \gamma u = \psi\;.$$

Proof . We have seen that under **A2** any solution $\chi = \binom{v_1}{v_2}$ of (6.10) or (6.11) solves the system

$$\begin{aligned} Av_1 - Bv_2 &= (D + I)f_1 - Sf_2 \\ B'v_1 + Cv_2 &= -Hf_1 - (D' - I)f_2\;. \end{aligned} \tag{6.12}$$

Now Corollary 3.2 and (6.4) imply that $\chi \in V_2^{\perp}$, hence due to Corollary 5.1 the solution is uniquely determined. ■

Theorem 6.2. *Suppose that Γ satisfies the assumption* **A1**. *Then for any $\psi = \binom{f_1}{f_2} \in V(\Gamma)$ the systems of boundary integral equations*

$$\begin{aligned} Av_1 - Bv_2 &= (D-I)f_1 - Sf_2 \\ B'v_1 + Cv_2 &= -Hf_1 - (D'+I)f_2 \, . \end{aligned} \tag{6.13}$$

and

$$\begin{aligned} Av_1 - Bv_2 &= (D-I)f_1 - Sf_2 \\ Sv_1 - (D-I)v_2 &= 0 \end{aligned} \tag{6.14}$$

are uniquely solvable. The solution $\chi = \binom{v_1}{v_2} \in (V(\Gamma))'$ coincides with the Neumann data δu of the function $u \in L_2$ solving the exterior Dirichlet problem (5.1) *with $\gamma u = \psi$. If* **A1** *is violated, then the systems* (6.13) *and* (6.14) *with vanishing right–hand side possess nontrivial solutions.*

Proof. We have seen that under **A1** the solution of (6.13) solves (6.14), too. Hence in view of Theorem 5.1 it suffices to prove that (6.14) is uniquely solvable. To this end we show that the second equation of this system determines $V_1^\perp$ even if the assumption **A2** is violated.
Indeed, in this case the Robin potential ρ spans the kernel of the operator S and we have $\binom{0}{\rho} = \gamma w$ with a nontrivial solution w of the homogeneous Dirichlet problem for Laplace's equation in Ω_2 satisfying the radiation condition (5.8). Note that in general $\gamma w \notin V(\Gamma)$, this requires a smoother curve, say $\Gamma \in C^{1,\alpha}$. But the vector $\binom{\rho}{0} \in (V(\Gamma))'$, in view of Corollary 5.2 we obtain that $\binom{\rho}{0} \in V_1^\perp$ and consequently

$$\chi = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in V_1^\perp \quad \Longleftrightarrow \quad Sv_1 - (D-I)\, v_2 = 0 \, . \tag{6.15}$$

Thus the solution $\chi = \binom{v_1}{v_2}$ of the homogeneous system (6.14) belongs to $V_1^\perp$, from Corollary 5.3 we conclude that $\mathcal{K}_0\chi \in H^2(\Omega_1)$ is harmonic. But the first equation of this system requires $\mathcal{K}_0\chi|_\Gamma = 0$ such that $\mathcal{K}_0\chi = 0$ in Ω_1 and $\mathcal{A}\chi = 0$. Hence, the homogeneous system (6.14) has a nontrivial solution only if the assumption **A1** is violated. ■

The previous result gives a necessary and sufficient condition on Γ to derive equivalent boundary integral equations for the exterior Dirichlet problem, whereas Theorem 6.1 contains only a sufficient condition for the interior Dirichlet problem. We can prove that the assumption **A2** is also necessary for the unique solvability of the systems of integral equations if Γ is sufficiently smooth.

Theorem 6.3. *Let $\Gamma \in C^{1,\alpha}$, $0 < \alpha < 1$, $\operatorname{cap} \Gamma = e^{-1}$ and $f_1 = f_2 = 0$. Then the systems* (6.10) *and* (6.11) *possess nontrivial solutions.*
Proof. We construct the nontrivial solution of (6.10) following a method in Fuglede [13]. Since *cap* $\Gamma = e^{-1}$ there exists a function w harmonic in Ω_2, satisfying the radiation condition (5.8) such that $w|_\Gamma = 0$ and $\gamma w \neq 0$. The condition $\Gamma \in C^{1,\alpha}$ ensures $\gamma w \in V(\Gamma)$ such that the solution u of the Dirichlet problem

$$\Delta^2 u = 0 \quad \text{in } \Omega_1 \, , \quad \gamma u = \gamma w \, ,$$

provides $0 \neq \delta u \in V_2^\perp$ and $\mathcal{A}\delta u = 2\mathcal{P}_1\gamma w = 2\gamma w$. Hence $\delta u \in (V(\Gamma))'$ solves the homogeneous system (6.10).

To get the nontrivial solution of (6.11) we start with $\mathcal{K}_0\binom{\rho}{0} \in H^2(\Omega_1)$ (see the proof of Theorem 6.2) and denote $\mathcal{A}\binom{\rho}{0} = 2\gamma\mathcal{K}_0\binom{\rho}{0} = 2\binom{w_1}{w_2}$. Then we solve the Neumann problem for the Laplace equation

$$\Delta v = 0 \quad \text{in } \Omega_2 \, , \quad \partial_n v|_\Gamma = -w_2 \, , \quad v \text{ satisfies } (5.8) \, .$$

It is well known that $\Gamma \in C^{1,\alpha}$ implies $\gamma v \in V(\Gamma)$, hence the solution of the Dirichlet problem

$$\Delta^2 u = 0 \quad \text{in } \Omega_1 \, , \quad \gamma u = \gamma v \, ,$$

gives $\delta u \in V_2^\perp$ with $\mathcal{A}\delta u = 2\mathcal{P}_1\gamma v = 2\gamma v$. So we derive

$$\mathcal{A}\Big(\delta u + \begin{pmatrix}\rho\\0\end{pmatrix}\Big) = 2\begin{pmatrix}v|_\Gamma + w_1\\ 0\end{pmatrix} .$$

Now we use that **A2** is violated. Then $S\rho = 0$ and for $\chi = \binom{v_1}{v_2} \in (V(\Gamma))'$ we obviously obtain the relation

$$Sv_1 - (D+I)\,v_2 = 0 \quad \Longleftrightarrow \quad \chi \in V_2^\perp \dot{+} \operatorname{span}\Big\{\begin{pmatrix}\rho\\0\end{pmatrix}\Big\} . \tag{6.16}$$

Thus $\delta u + \binom{\rho}{0}$ is a nontrivial solution of the homogeneous system (6.11). ■

Remark 6.1. The system (6.10) with the operator S replaced by the usual weakly singular operator

$$S_1\varphi(x) := -\frac{1}{\pi}\int_\Gamma \ln|x-y|\,\varphi(y)\,ds_y$$

was introduced in [4] and analysed in [13] for the case that the data satisfy the conditions $\Gamma \in C^{1,\alpha}$, $f_1 \in C^1(\Gamma)$, $f_2 \in C(\Gamma)$. It was proved that the corresponding integral equations are uniquely solvable and provide the solution of the interior Dirichlet problem iff *cap* $\Gamma \notin \{e^{-1}, 1\}$. It can be easily seen that under this assumption the assertions of Theorems 6.1 and 6.3 remain true for the systems (6.10) and (6.11) with the operator S_1 instead of S.

Finally we mention the problem to describe the assumption **A1** in terms of the boundary Γ. If Γ is a circle of radius r then a straightforward calculation shows that the homogeneous system $\mathcal{A}\chi = 0$ has a nontrivial solution if and only if $r = e^{-1}$. Therefore it was conjectured in [9] that in general **A1** is valid iff *cap* $\Gamma \neq e^{-1}$, i.e. coincides with our assumption **A2**. We tried to verify the conjecture by using the fact that **A1** and **A2** are the solvability conditions of integral equations for the same Dirichlet problem, but unsuccessfully.

By Corollary 5.1 the kernel of the operator $\mathcal{A}$ has an empty intersection with $l\,(\Gamma)^\perp \subset (V(\Gamma))'$. We conjecture that $\chi \in \ker \mathcal{A}$ implies $I_1\chi = 0$, such that there exist at most two linear independent solutions of the exterior Dirichlet problem having zero trace and satisfying the radiation condition (4.3).

In the special case of the circle with radius $r = e^{-1}$ we have the following situation. The kernel of $\mathcal{A}$ is spanned by of the two vectors $(e\,x_k,\, -x_k) \in V_1^\perp$, $k = 1, 2$, and the corresponding solutions of the homogeneous Dirichlet problem for the exterior of this circle are the functions

$$u_k(x_1, x_2) = x_k\big(2\ln|x| + 1 + e^{-2}\,|x|^{-2}\big) .$$

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